FOURIER-Transformation

f-Form

$$X(\mathrm{j}f) = \int_{-\infty}^{+\infty} x(t)\cdot \mathrm{e}^{-\mathrm{j}2\pi f t}\,\mathrm{d}t$$

$$x(t) = \int_{-\infty}^{+\infty} X(\mathrm{j}f)\cdot \mathrm{e}^{\mathrm{j}2\pi f t}\,\mathrm{d}f$$

formaler Übergang

$$f = \frac{\omega}{2\pi} \longrightarrow \mathrm{d}f = \frac{\mathrm{d}\omega}{2\pi}$$

ω-Form

$$X(\mathrm{j}\omega) = \int_{-\infty}^{+\infty} x(t)\cdot \mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\mathrm{j}\omega)\cdot \mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega$$

Die Korrespondenzen für beide Formen sind identisch, mit Ausnahme darin enthaltener DIRAC-Stöße!
Letztere sind zu ersetzen: $\delta(f) \to 2\pi\delta(\omega)$

Komponentenform:

$$x(t) = x_\mathrm{g}(t) + x_\mathrm{u}(t) \text{ mit } x_\mathrm{g}(t) = \frac{x(t)+x(-t)}{2} \text{ sowie } x_\mathrm{u}(t) = \frac{x(t)-x(-t)}{2}$$

$$X(\mathrm{j}\omega) = X_\mathrm{g}(\mathrm{j}\omega) + X_\mathrm{u}(\mathrm{j}\omega) = 2\int_0^\infty x_\mathrm{g}(t)\cos(\omega t)\mathrm{d}t - \mathrm{j}2\int_0^\infty x_\mathrm{u}(t)\sin(\omega t)\mathrm{d}t$$

$$x(t) = x_\mathrm{g}(t) + x_\mathrm{u}(t) = \frac{1}{\pi}\int_0^\infty X_\mathrm{g}(\mathrm{j}\omega)\cos(\omega t)\mathrm{d}\omega - \frac{1}{\pi}\int_0^\infty X_\mathrm{u}(\mathrm{j}\omega)\sin(\omega t)\mathrm{d}\omega$$

FOURIER-Transformation (ω-Form)

$\int x^2(t)\,\mathrm{d}t < \infty$:

$$X(\mathrm{j}\omega) = \int_{-\infty}^{+\infty} x(t)\cdot \mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t$$

Spektralfunktion
(FOURIER-Transformierte)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\mathrm{j}\omega)\cdot \mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega$$

Zeitfunktion
(inverse FOURIER-Transf.)

$$X(\mathrm{j}\omega) = \mathfrak{F}\{x(t)\}$$
$$x(t) = \mathfrak{F}^{-1}\{X(\mathrm{j}\omega)\}$$

Zeitraum **Frequenzraum**

$$x(t) \circ\!\!-\!\!\bullet\, X(\mathrm{j}\omega)$$

FOURIER-Korrespondenz

Einheiten: $[x(t)]$ = Amplitude; $[X(\mathrm{j}\omega)]$ = Amplitude/Frequenz

Einige Korrespondenzen der δ-Funktion

a) $X(\mathrm{j}\omega) = \int_{-\infty}^{+\infty} \delta(t) \cdot \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{d}t = \int_{-\infty}^{+\infty} \delta(t) \cdot \mathrm{e}^{-\mathrm{j}\omega 0} \mathrm{d}t = \mathrm{e}^{-\mathrm{j}\omega 0} \int_{-\infty}^{+\infty} \delta(t) \mathrm{d}t = \mathrm{e}^{-\mathrm{j}\omega 0} = 1 \quad \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 1 \cdot \mathrm{e}^{\mathrm{j}\omega t} \mathrm{d}\omega$

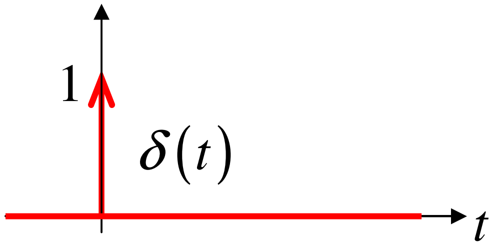
b) $X(\mathrm{j}\omega) = \int_{-\infty}^{+\infty} \delta(t - t_0) \cdot \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{d}t = 1 \cdot \mathrm{e}^{-\mathrm{j}\omega t_0} \qquad \delta(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{j}\omega(t - t_0)} \mathrm{d}\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{j}\omega t_0} \cdot \mathrm{e}^{\mathrm{j}\omega t} \mathrm{d}\omega$

c) $x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \delta(\omega) \cdot \mathrm{e}^{\mathrm{j}\omega t} \mathrm{d}\omega = \frac{1}{2\pi}$

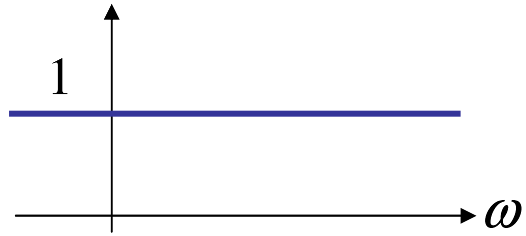
Probe: $X(\mathrm{j}\omega) = \int_{-\infty}^{+\infty} \frac{1}{2\pi} \cdot \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{d}t = \frac{\cancel{2}\cancel{\pi}\,\delta(-\omega)}{\cancel{2}\cancel{\pi}} = \delta(\omega)$

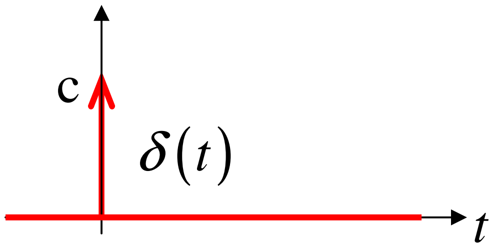
d) $x(t) = 1 \quad \circ\!\!-\!\!\bullet \quad X(\mathrm{j}\omega) = 2\pi\delta(\omega)$

Anschaulich

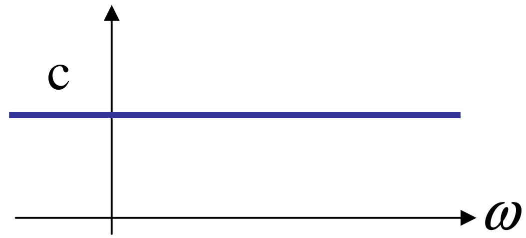


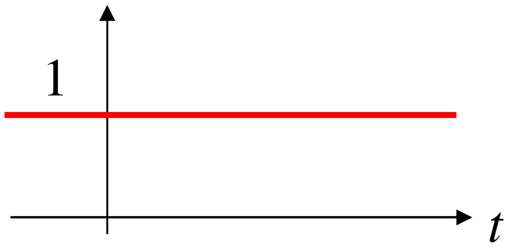
$$\delta(t) \circ\!\!-\!\!\bullet\ 1$$



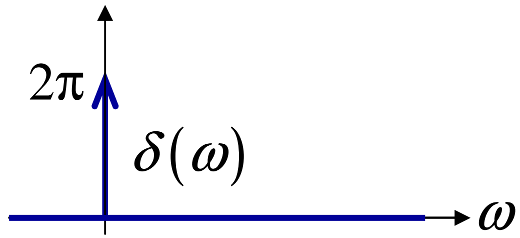


$$c \cdot \delta(t) \circ\!\!-\!\!\bullet\ c$$





$$1 \circ\!\!-\!\!\bullet\ 2\pi\,\delta(\omega)$$

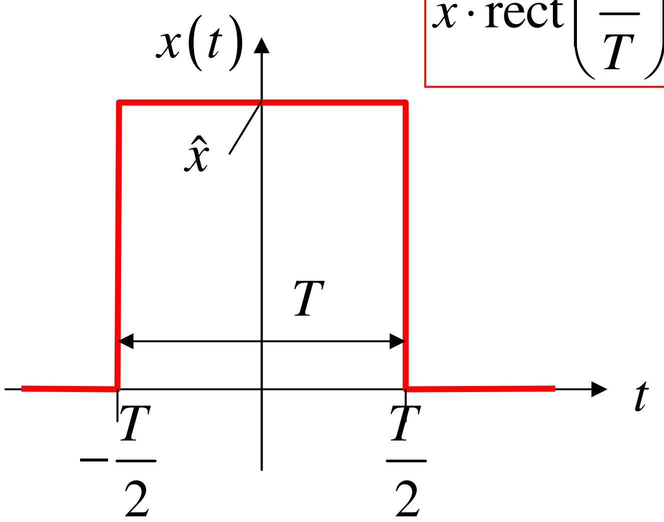


eine weitere Korrespondenz

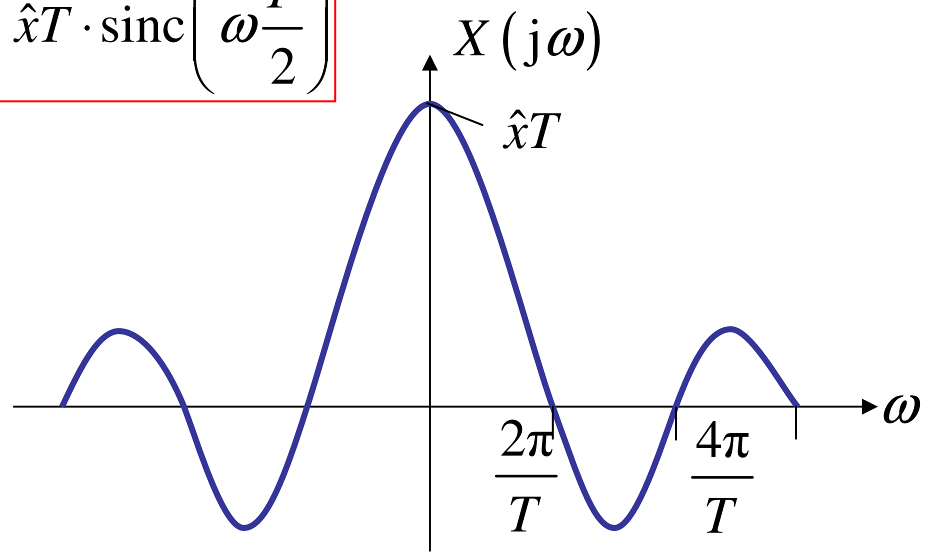
$$X(\mathrm{j}\omega) = \int_{-\infty}^{+\infty} \hat{x}\cdot \mathrm{rect}\left(\frac{t}{T}\right)\cdot \mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t = \hat{x}\int_{-\frac{T}{2}}^{+\frac{T}{2}} \mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t = \frac{\hat{x}}{-\mathrm{j}\omega}\left(\mathrm{e}^{-\mathrm{j}\omega\frac{T}{2}} - \mathrm{e}^{+\mathrm{j}\omega\frac{T}{2}}\right)$$

$$= \frac{\hat{x}\cdot\left(-2\not{\mathrm{j}}\cdot\sin\left(\omega\frac{T}{2}\right)\right)}{-\not{\mathrm{j}}\omega} = \frac{2\hat{x}T\cdot\sin\left(\omega\frac{T}{2}\right)}{\omega T} = \hat{x}T\cdot\mathrm{sinc}\left(\omega\frac{T}{2}\right)$$

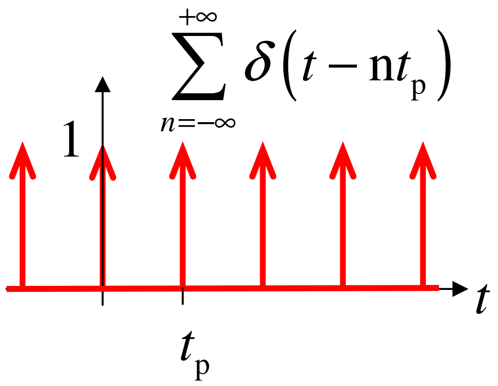
$$\hat{x}\cdot\mathrm{rect}\left(\frac{t}{T}\right) \quad \circ\!\!-\!\!\bullet \quad \hat{x}T\cdot\mathrm{sinc}\left(\omega\frac{T}{2}\right)$$

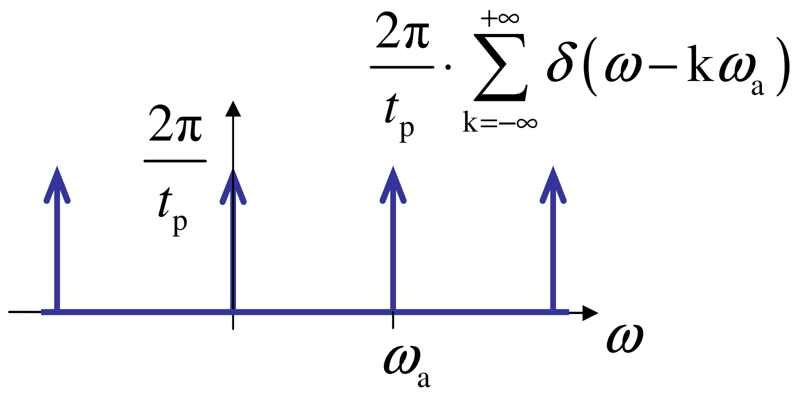


$\circ\!\!-\!\!\bullet$



Impulskamm (Stoßfolge)





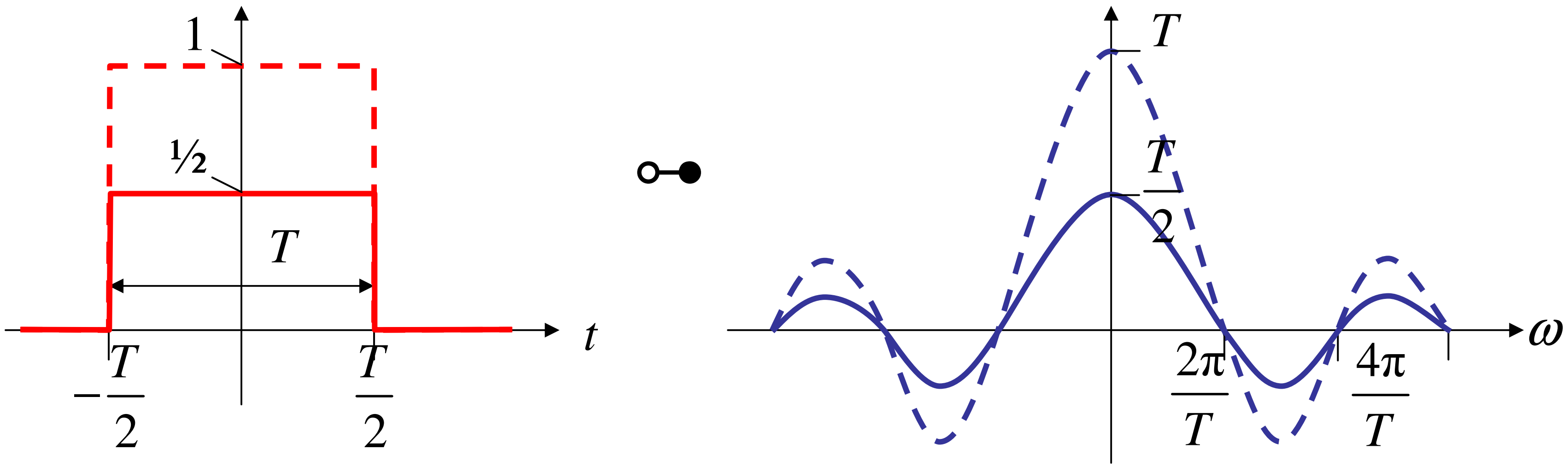
$$\text{Ш}(t - nt_p) \circ\!\!-\!\!\bullet \; \omega_a \cdot \text{Ш}(\omega - k\omega_a)$$

$$\omega_a = \frac{2\pi}{t_p}$$

Eigenschaften

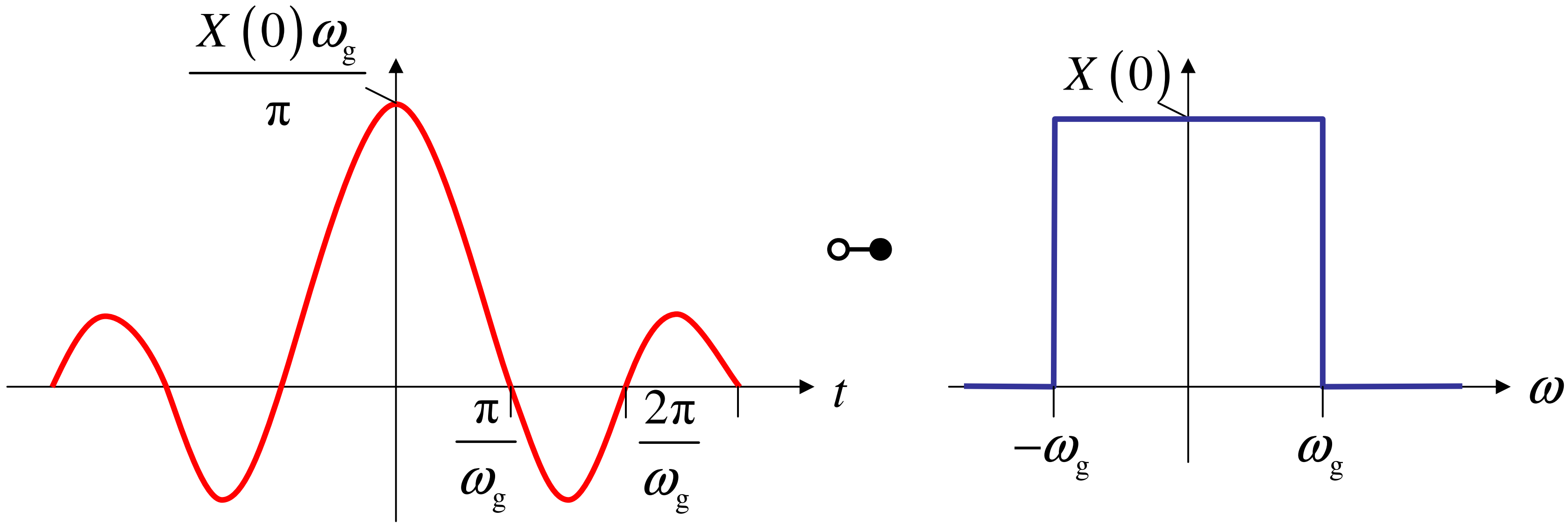
<u>Voraussetzung:</u> $x_1(t) \circ\!\!-\!\!\bullet X_1(j\omega)$ und $x_2(t) \circ\!\!-\!\!\bullet X_2(j\omega)$

Linearitätssatz: $\lambda_1 x_1(t) + \lambda_2 x_2(t) \circ\!\!-\!\!\bullet \lambda_1 X_1(j\omega) + \lambda_2 X_2(j\omega)$



Vertauschungssatz

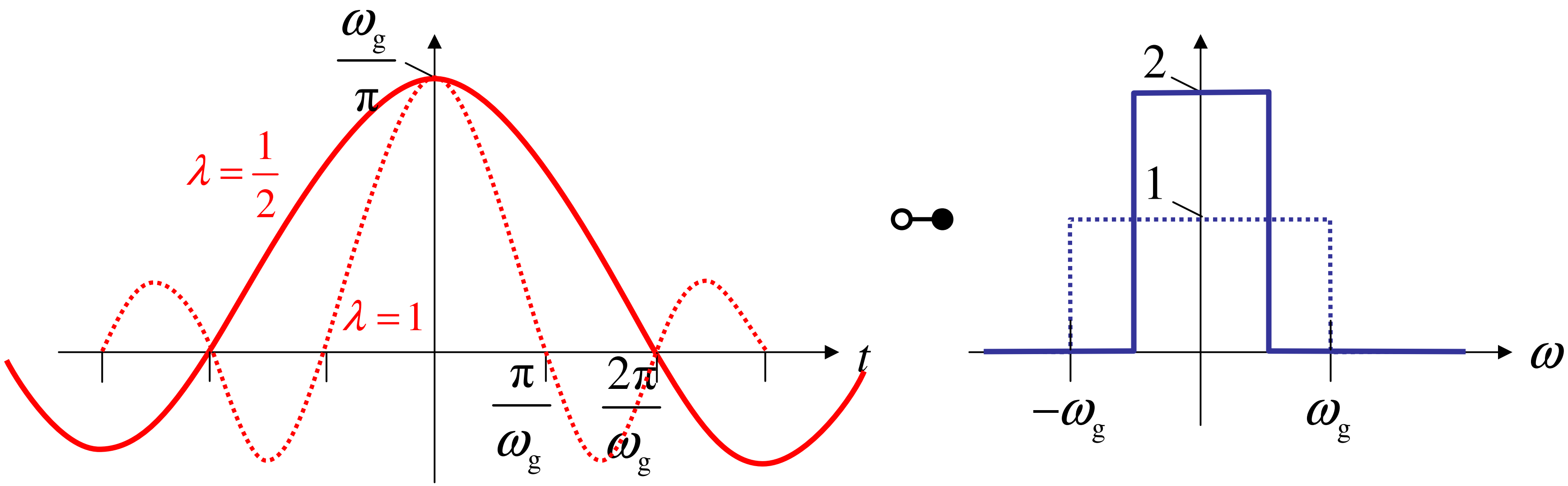
$$x(t) \circ\!\!-\!\!\bullet X(j\omega) \quad \rightarrow \quad X(t) \circ\!\!-\!\!\bullet 2\pi x(-j\omega)$$



$$\frac{X(0)\,\omega_g}{\pi}\cdot \mathrm{sinc}(\omega_g t) \circ\!\!-\!\!\bullet X(0)\cdot \mathrm{rect}\left(\frac{\omega}{2\omega_g}\right)$$

Ähnlichkeitssatz

$$x(\lambda t) \circ\!\!-\!\!\bullet \frac{1}{|\lambda|} X\left(\frac{j\omega}{\lambda}\right)$$

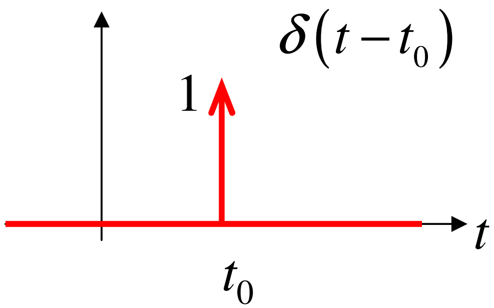


$$\frac{\omega_g}{\pi} \cdot \mathrm{sinc}\left(\frac{1}{2}\omega_g t\right) \circ\!\!-\!\!\bullet 2 \cdot \mathrm{rect}\left(\frac{\omega}{\omega_g}\right)$$

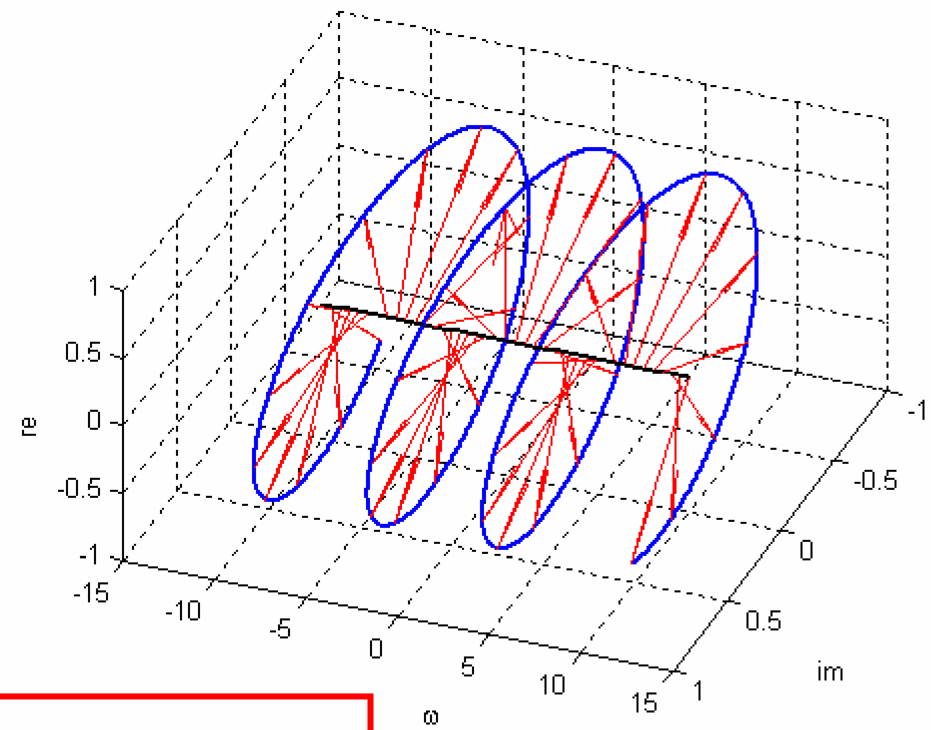
Zeitverschiebungssatz

$$x(t - t_0) \;\circ\!\!-\!\!\bullet\; X(\mathrm{j}\omega)\cdot \mathrm{e}^{-\mathrm{j}\omega t_0}$$

Beispiel:



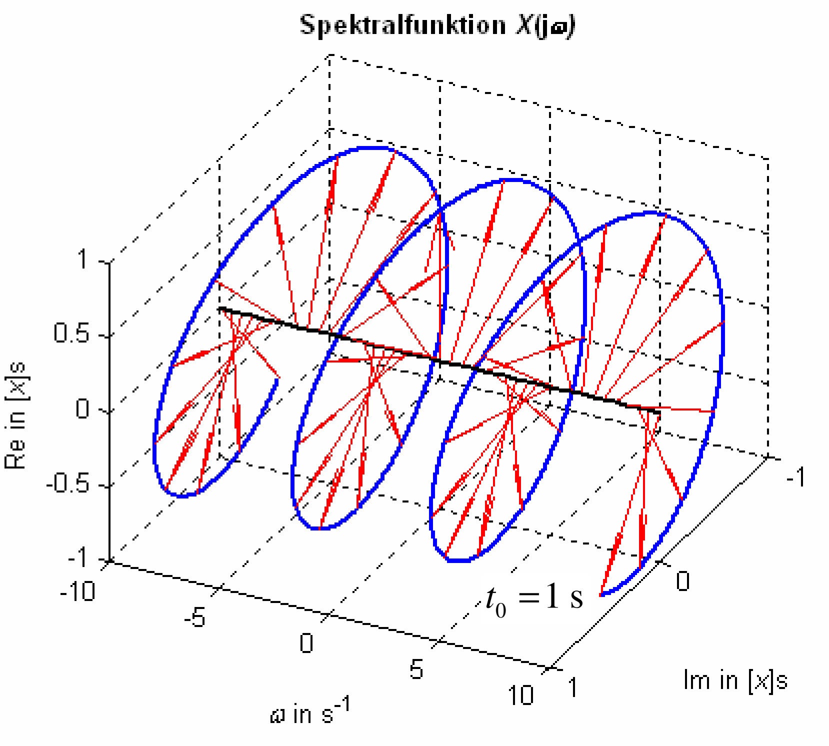
$\circ\!\!-\!\!\bullet$



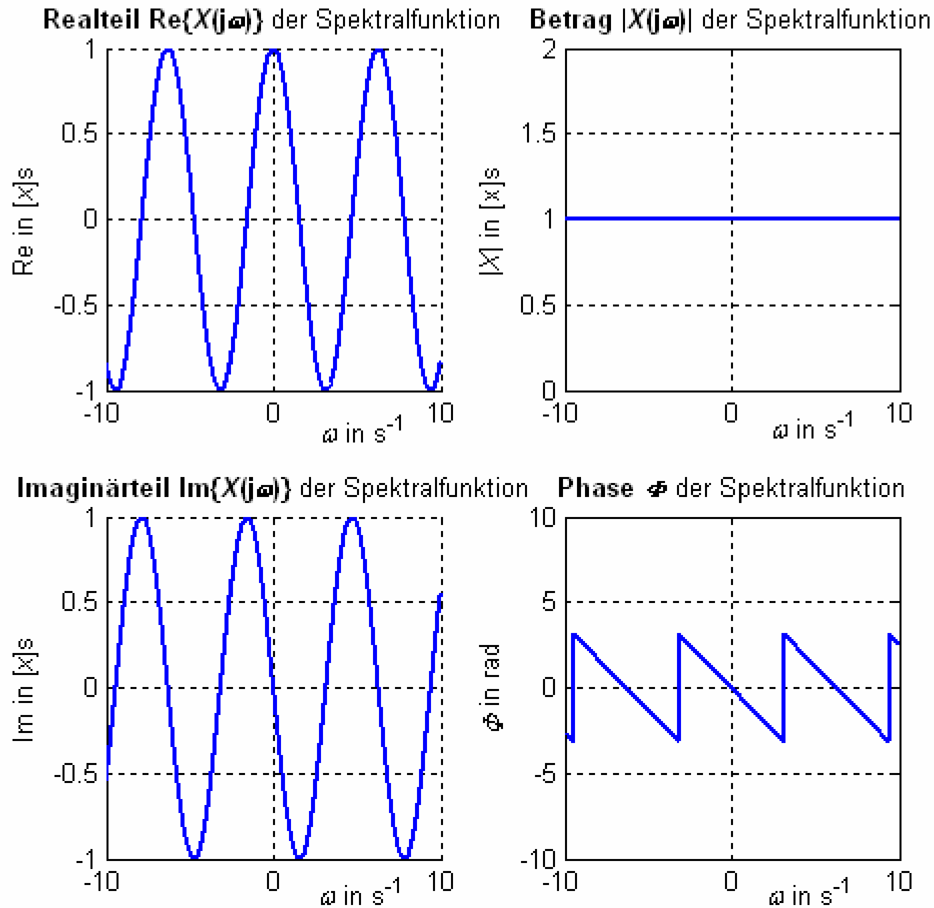
$$\delta(t-t_0) \;\circ\!\!-\!\!\bullet\; 1\cdot \mathrm{e}^{-\mathrm{j}\omega t_0}$$

Spektralfunktion des verschobenen δ-Stoßes

$$\delta(t - t_0) \circ\!\!-\!\!\bullet \quad 1 \cdot \mathrm{e}^{-\mathrm{j}\omega t_0}$$



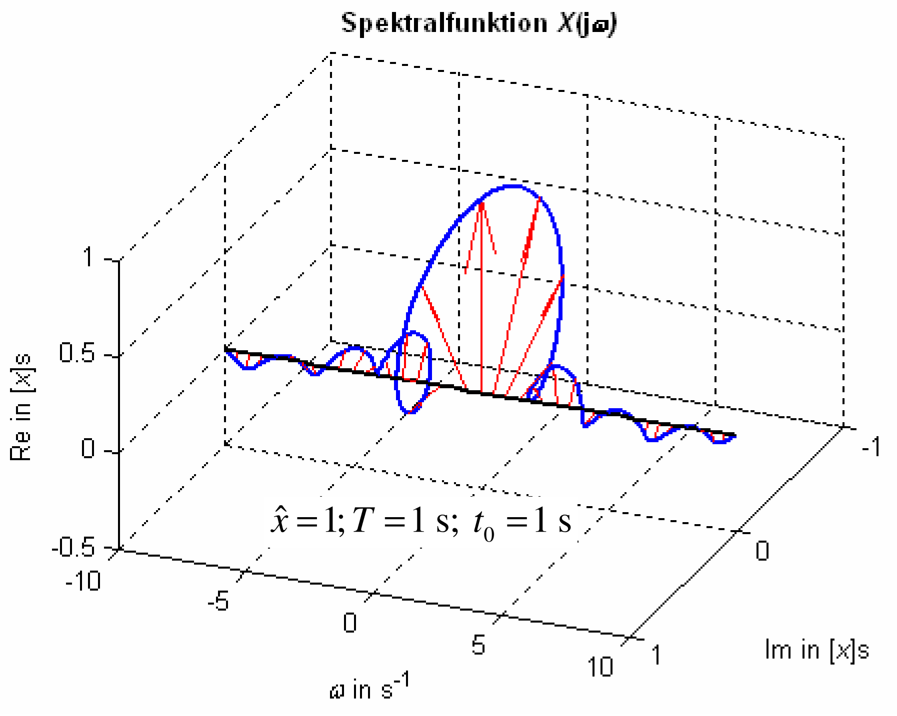
3D-Spektralfunktion



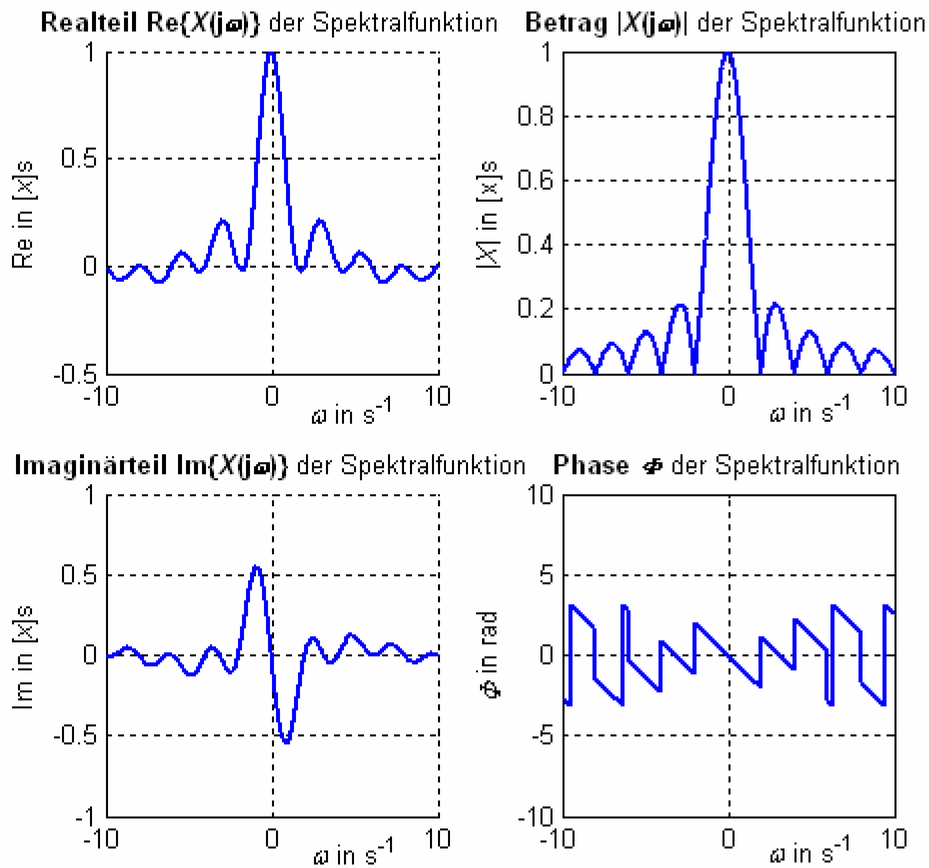
"Seitenansichten"

Spektralfunktion der verschobenen rect-Funktion

$$\hat{x} \cdot \mathrm{rect}\left(\frac{t - t_0}{T}\right) \circ\!\!-\!\!\bullet \quad \hat{x}T \cdot \mathrm{sinc}\left(\omega \frac{T}{2}\right) \cdot \mathrm{e}^{-\mathrm{j}\omega t_0}$$

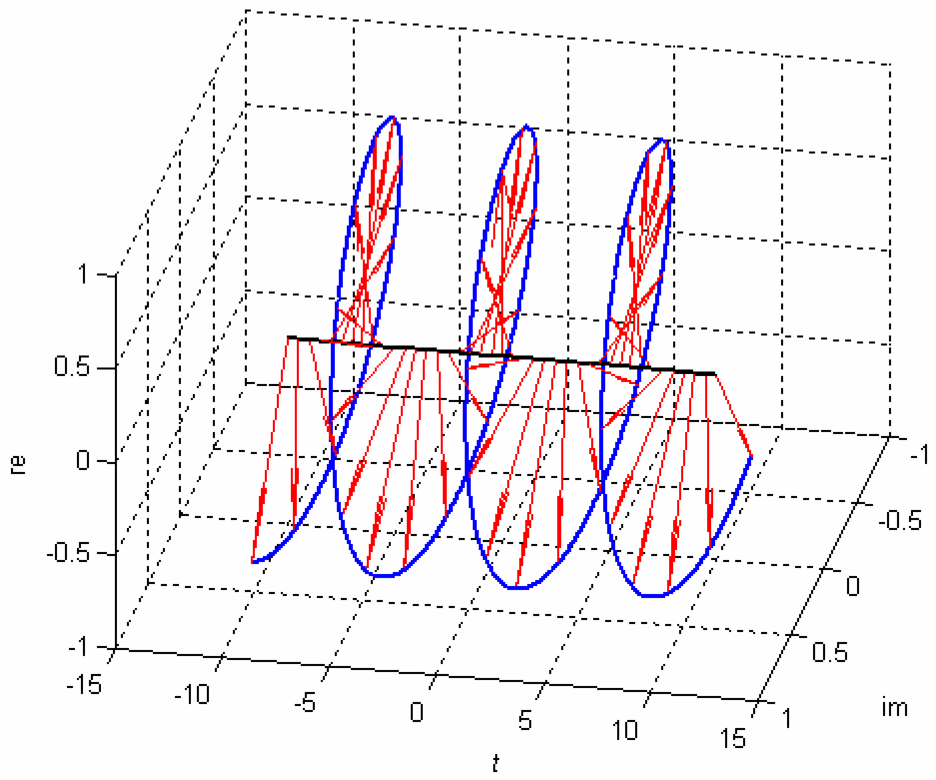


3D-Spektralfunktion

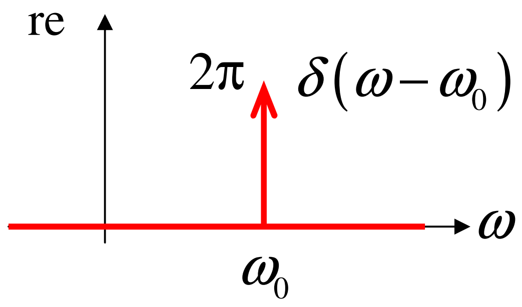


Frequenzverschiebungssatz

$$x(t)\cdot e^{j\omega_0 t} \quad ○\!\!-\!\!\bullet \quad X(j(\omega-\omega_0))$$



○—●

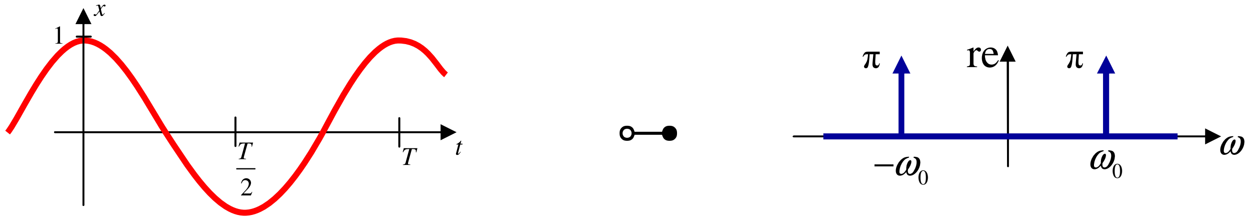


$$e^{j\omega_0 t} \; ○\!\!-\!\!\bullet \; 2\pi\delta(\omega-\omega_0)$$

Modulationssatz

...und mit Linearitätssatz folgt daraus:

$$\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}\omega_0 t} + \mathrm{e}^{-\mathrm{j}\omega_0 t}\right) = \cos\left(\omega_0 t\right) \circ\!\!-\!\!\bullet\ \pi\,\{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\}$$

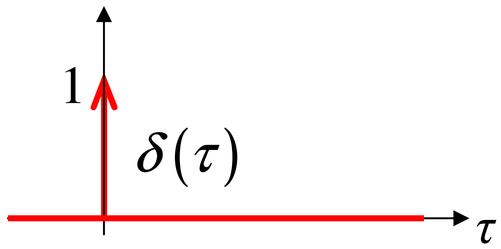


...oder allgemein:

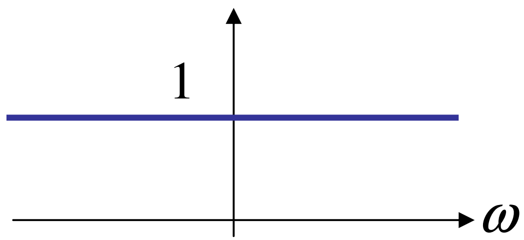
$$x(t)\cdot\cos\left(\omega_0 t\right)\ \circ\!\!-\!\!\bullet\ \pi\,\{X(\mathrm{j}(\omega - \omega_0)) + X(\mathrm{j}(\omega + \omega_0))\}$$

Integrationssatz

$$\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau \;\circ\!\!-\!\!\bullet\; X(\mathrm{j}\omega)\left(\pi\delta(\omega)+\frac{1}{\mathrm{j}\omega}\right)$$



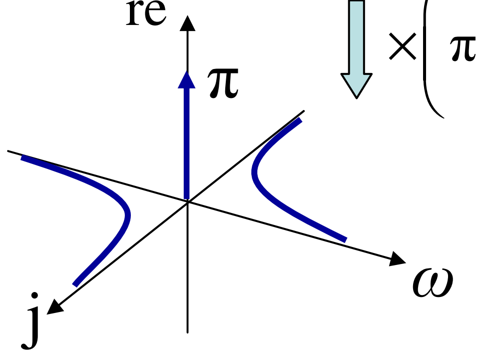
$\circ\!\!-\!\!\bullet$



$\int \ldots \mathrm{d}\tau$ ⇓

⇓ $\times\left(\pi\delta(\omega)+\frac{1}{\mathrm{j}\omega}\right)$

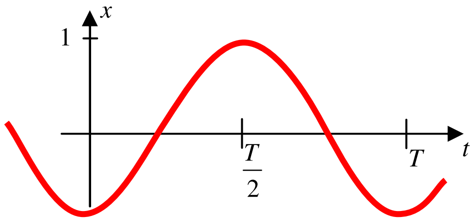
$\circ\!\!-\!\!\bullet$



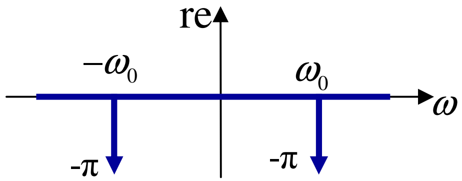
$$\sigma(t)=\int_{-\infty}^{t}\delta(\tau)\,\mathrm{d}\tau \;\circ\!\!-\!\!\bullet\; \pi\delta(\omega)+\frac{1}{\mathrm{j}\omega}$$

Differentationssatz

$$x^{(n)}(t) \circ\!\!-\!\!\bullet\, (j\omega)^n X(j\omega)$$



$\circ\!\!-\!\!\bullet$

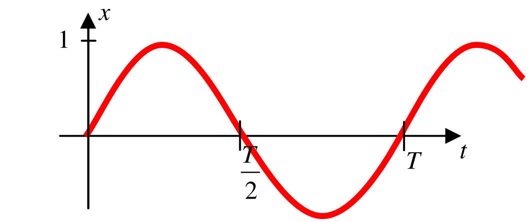


$\dfrac{d}{dt}\Big\downarrow$

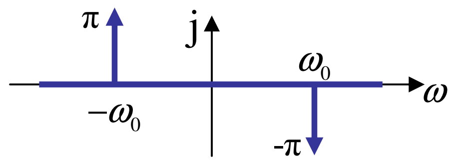
$$\frac{d}{dt}\left[-\cos\left(\omega_0 t\right)\right] = \omega_0 \cdot \sin\left(\omega_0 t\right) \quad \circ\!\!-\!\!\bullet \quad \left[-\pi\left\{\delta\left(\omega - \omega_0\right) + \delta\left(\omega + \omega_0\right)\right\}\right] \cdot j\omega =$$

$$-j\pi\left\{\delta\left(\omega - \omega_0\right)\omega_0 + \delta\left(\omega + \omega_0\right)\left(-\omega_0\right)\right\}$$

$\Big\downarrow \times j\omega$



$\circ\!\!-\!\!\bullet$



2. Bsp.:

$$\frac{d}{dt}\sigma(t) = \sigma^{(1)}(t) = \delta(t) \quad \circ\!\!-\!\!\bullet \quad j\omega \cdot \left(\pi\delta(\omega) + \frac{1}{j\omega}\right) = 1$$

Faltungssatz

$$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau) \cdot x_2(t-\tau)\,\mathrm{d}\tau \;\circ\!\!-\!\!\bullet\; X_1(\mathrm{j}\omega) \cdot X_2(\mathrm{j}\omega)$$

und umgekehrt:

$$2\pi \cdot x_1(t) \cdot x_2(t) \;\circ\!\!-\!\!\bullet\; X_1(\mathrm{j}\omega) * X_2(\mathrm{j}\omega)$$

bzw.

$$x_1(t) \cdot x_2(t) \;\circ\!\!-\!\!\bullet\; \frac{1}{2\pi} \cdot X_1(\mathrm{j}\omega) * X_2(\mathrm{j}\omega)$$

WICHTIG: Die Faltung im Frequenzraum erfolgt mit den 3D-Spektren!

Beispiel zum Faltungssatz

Bestätigung des Additionstheorems $\sin^2\alpha = \frac{1}{2} - \frac{1}{2}\cos(2\alpha)$ mittels FOURIER-Theorie!

$$\alpha = \omega_0 t \;\Rightarrow\; \sin^2(\omega_0 t) = \sin(\omega_0 t)\cdot\sin(\omega_0 t)$$