Übungsaufgaben Analysis II

KS

1.) a) $f(x,y) = x^2 \ln(y) + y^2 \ln(x) + xy + 7$

$f_x = 2x \ln(y) + y^2 \frac{1}{x} + y$ | NR:

$f_y = \frac{x^2}{y} + 2y \ln(x) + x$ | $x^2 \ln(y)$

$f_{xx} = 2\ln(y) + \left(-\frac{y^2}{x^2}\right) + 0$

$f_{yy} = -\frac{x^2}{y^2} + 2\ln(x) + 0$

$f_{xy} = \frac{2x}{y} + \frac{2y}{x} + 1$

b) $f(x,y) = e^{x+y^2} x^2 + \ln(x^2) + y$

$f_x = e^{x+y^2} \cdot x^2 + e^{x+y^2} \cdot 2x + \frac{2}{x}$

$f_y = e^{x+y^2} \cdot x^2 \cdot 2y + 1$

~~$f_{xx} = e^{x+y^2} \cdot x^2 + e^{x+y^2} \cdot 2x$~~

$f_x = e^{x+y^2} (x^2 + 2x) + \frac{2}{x}$

$f_{xx} = e^{x+y^2} (x^2 + 2x) + e^{x+y^2} (2x + 2) + \left(-\frac{2}{x^2}\right)$

$f_{xx} = e^{x+y^2} (x^2 + 4x + 2) - \frac{2}{x^2}$

$f_{yy} = e^{x+y^2} \cdot 2y \cdot 2yx^2 + e^{x+y^2} \cdot 2x^2$

$f_{yy} = 2x^2 \left(e^{x+y^2} \cdot 2y \cdot y + e^{x+y^2}\right)$

$f_{yy} = 2x^2 \left(e^{x+y^2} \cdot 2y^2 + e^{x+y^2}\right)$

$f_{xy} = e^{x+y^2} (2x^2 y + 4xy)$

c) $f(x,y) = (x+7)^3 (y+1)^2$

$f_x = (y+1)^2 \cdot [3(x+7)^2 \cdot 1]$

$f_x = 3(x+7)^2 (y+1)^2$

$f_y = 2(y+1)(x+7)^3$

$f_{xx} = 6(x+7)(y+1)^2$

$f_{yy} = 2(x+7)^3$

$f_{xy} = 6(x+7)^2 (y+1)$

d) $f(x,y) = \frac{x}{y^2} + x^4 e^y$

$f_x = \frac{1}{y^2} + 4x^3 e^y$

$f_y = -2\frac{x}{y^3} + x^4 e^y$

$f_{xx} = 12x^2 e^y$ $\qquad f_{yy} = 6\frac{x}{y^4} + x^4 e^y$

$f_{xy} = -2\frac{1}{y^3} + 4x^3 e^y$

e)

$$f(x,y,z) = x^3yz^2 + 2x^2y^2z + 5z^3$$

$$f_x = 3x^2yz^2 + 4xy^2z + 0$$

$$f_y = x^3z^2 + 4x^2yz$$

$$f_z = 2x^3yz + 2x^2y^2 + 15z^2$$

$$f_{xx} = 6xyz^2 + 4y^2z$$

$$f_{yy} = \cancel{3x^2z^2} \; 4x^2z$$

$$f_{zz} = 2x^3y + 30z$$

$$f_{xxx} = 6yz^2$$

$$f_{yyy} = 0$$

$$f_{zzz} = 30$$

f) $f(x,y) = \sqrt{y-x^2}$

$f_x = \frac{1}{2}(y-x^2)^{-\frac{1}{2}} \cdot (-2x)$

$f_x = -\frac{2x}{2(y-x^2)^{\frac{1}{2}}} = -\frac{x}{\sqrt{y-x^2}}$

$f_y = \frac{1}{2}(y-x^2)^{-\frac{1}{2}} = \frac{1}{2\sqrt{y-x^2}}$

~~$f_{xx} = -\frac{1}{4}$~~

$u = \frac{1}{2}(y-x^2)^{-\frac{1}{2}}$

$u' = -\frac{1}{4}(y-x^2)^{-\frac{3}{2}} \cdot (-2x)$ *

$v = (-2x)$

$v' = -2$

$(uv)' = u'v + uv'$

$f_x = -\frac{1}{4}(y-x^2)^{-\frac{3}{2}} \cdot (-2x)(-2x) + \frac{1}{2}(y-x^2)^{-\frac{1}{2}} \cdot -2$

$f_x = -\frac{x^2}{\sqrt{(y-x^2)^3}} + \left(-\frac{1}{\sqrt{y-x^2}}\right)$

$f_x =$

$u = -x$

$u' = +1$

$v = -\sqrt{y-x^2}$

$v' = -\frac{1}{2}(y-x^2)^{-\frac{1}{2}} \cdot (-2x)$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$f_{xx} = \frac{+\sqrt{y-x^2} - \frac{-x \cdot (-2x)}{2\sqrt{y-x^2}}}{y-x^2}$$

$$f_{xx} = \frac{\sqrt{y-x^2} - \frac{x^2}{\cancel{4}\sqrt{y-x^2}}}{y-x^2}$$

$$f_{yy} = \left[\frac{1}{2}(y-x^2)^{-\frac{1}{2}}\right]'$$

$$= -\frac{1}{4}(y-x^2)^{-\frac{3}{2}} \cdot 1$$

$$= -\frac{1}{4 \cdot \sqrt{(y-x^2)^3}}$$

g) $f(x,y) = \dfrac{x^2 - y}{x}$

$f_x = \left[\dfrac{x^2-y}{x} \right]'$

$u = x^2 - y$

$u' = 2x$

$v = \frac{1}{x}$

$v' = -\frac{1}{x^2}$

$\left(\frac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$

$f_x = \dfrac{2x \cdot \frac{1}{x} - (x^2-y) \cdot \left(-\frac{1}{x^2}\right)}{x^2}$

$\cancel{f_x = \dfrac{2x^2 - x^2 + y}{x^2}} = \cancel{\dfrac{x^2+y}{x^2}}$

$f_x = \dfrac{2 - \left(1 + \frac{y}{x^2}\right)}{\left(\frac{1}{x}\right)^2}$

9/ $f(x,y) = \frac{x^2-y}{x} = \underbrace{(x^2-y)}_{u} \cdot \underbrace{x^{-1}}_{v}$

$(u \cdot v)' = uv' + u'v$

$u = x^2 - y$

$u' = 2x$

$v = x^{-1}$

$v' = -x^{-2}$

$(x^2-y) \cdot (-x^{-2}) + \frac{2x}{x}$

$-\frac{x^2+y}{x^2} + 2$

$-1 + \frac{y}{x^2} + 2$

$-\frac{x^2}{x^2} + \frac{y}{x^2} + \frac{2x^2}{x^2}$

$\frac{-x^2+y+2x^2}{x^2}$

$\frac{x^2}{x^2} + \frac{y}{x^2}$

$\underline{f_x = 1 + yx^{-2}}$

$f(x,y) = \frac{x^2 - y}{x}$

$\cancel{f_y} = \frac{x^2}{x} - \frac{y}{x}$

$x - yx^{-1}$

$f_y = -\frac{1}{x}$

$f_x = 1 + yx^{-2}$

$f_{xx} = -2yx^{-3}$

$f_y = -x^{-1} \cdot y^0$

$f_{yy} = \cancel{x^{-2}}\ 0$

$f_{xy} = +x^{-2} \overset{?}{=} f_{yx} = x^{-2}$

4) $f(x,y) = (1 + x^2 - 2y^2)^2$

$f_x = 2(1+x^2-2y^2) \cdot 2x = 4x + 4x^3 - 8xy^2$

$f_y = 2(1+x^2-2y^2)(-4y) = -8y - 8yx^2 + 16y^3$

$f_{xx} = 4 + 12x^2 - 8y^2 = 4(1 + 3x^2 - 2y^2)$

$f_{yy} = -8 - 8x^2 + 48y^2$

$f_{xy} = -16xy$

2.)

a)

$z = \sqrt{x-y} + \ln\sqrt{xy}$

$u = (x-y)^{\frac{1}{2}}$

$u' = \frac{1}{2}(x-y)^{-\frac{1}{2}}$

$v = x-y$

$v' = 1$

$[u(v)]' = u' \cdot v'$

$\cancel{\frac{1}{2\sqrt{x-y}}} \cancel{= f_x}$

$\frac{1}{2\sqrt{x-y}}$

$\underline{f_x = \frac{1}{2\sqrt{x-y}} + \frac{1}{2x}}$

$[\ln(u)]' = \frac{1}{u} \cdot u'$

$\frac{1}{\sqrt{xy}} \cdot \frac{1}{2}(xy)^{-\frac{1}{2}} \cdot y$

$\frac{y}{2\sqrt{xy}\sqrt{xy}}$

$\frac{1}{2x}$

$$z = \sqrt{x-y} + \ln\sqrt{xy}$$

$$-\frac{1}{2}(x-y)^{-\frac{1}{2}} + \frac{1}{\sqrt{xy}} \cdot \frac{1}{2}(xy)^{-\frac{1}{2}} \cdot x$$

$$-\frac{1}{2\sqrt{x-y}} + \frac{x}{2\sqrt{xy}\cdot\sqrt{xy}}$$

$$-\frac{1}{2\sqrt{x-y}} + \frac{1}{2y}$$

$$dz = \left(\frac{1}{2\sqrt{x-y}} + \frac{1}{2x}\right)dx + \left(-\frac{1}{2\sqrt{x-y}} + \frac{1}{2y}\right)dy$$

$$dz \left(\frac{1}{2\sqrt{x-y}} + \frac{1}{2x}\right)dx - \left(\frac{1}{2\sqrt{x-y}} - \frac{1}{2y}\right)dy$$

b) $u = x^3 + xy^2 + 2yz^2$

$f_x = 3x^2 + y^2$

$f_y = 2xy + 2z^2$

$f_z = 4yz$

$du = (3x^2 + y^2)\,dx + (2xy + 2z^2)\,dy + (4yz)\,dz$

c) $z = \sin(x^2 + y^2)$

$f_x = \cos(x^2 + y^2) \cdot 2x$

$f_y = \cos(x^2 + y^2) \cdot 2y$

$dz = [\cos(x^2 + y^2) \cdot 2x]\,dx + [\cos(x^2 + y^2) \cdot 2y]\,dy$

3.) $R = \frac{R_1 R_2}{R_1 + R_2}$

$R_1 = 450\Omega \pm 2\Omega$

$R_2 = 150\Omega \pm 1\Omega$

$R = 112{,}5\Omega$

$$f_{R_1} = \frac{R_2}{R_1+R_2} - \frac{R_1 R_2}{(R_1+R_2)^2}$$

$$f_{R_2} = \frac{R_1}{R_1+R_2} - \frac{R_1 R_2}{(R_1+R_2)^2}$$

$$dR = \left[\frac{R_2}{R_1+R_2} - \frac{R_1 R_2}{(R_1+R_2)^2}\right] dR_1 + \left[\frac{R_1}{R_1+R_2} - \frac{R_1 R_2}{(R_1+R_2)^2}\right] dR_2$$

$$dR = (0{,}0625 \cdot 2) + (0{,}5625 \cdot 1)$$

$$\underline{dR = 0{,}6875\Omega}$$

$\cancel{dR = 0{,}006\bar{7}}$

$\cancel{\partial}$ $\delta R = \frac{0{,}6875\Omega}{112{,}5\Omega} \cdot 100\% \approx 0{,}6\bar{1}\%$

4.)

a) $z = \sqrt{x^2+y^2}$

$z_x = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x$

$z_y = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y$

$P_0(3; 4; 5)$ $\alpha = 45^0$

$$\frac{\partial z}{\partial \vec{a}} = z_x \cdot \cos\alpha + z_y \cdot \sin\alpha$$

$$\frac{\partial z}{\partial \vec{a}} = \frac{2x\cos\alpha + 2y\sin\alpha}{2\sqrt{x^2+y^2}} = \frac{x\cdot\cos\alpha + \sin\alpha\cdot y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial \vec{a}} = \frac{3\cdot\cos(45^0) + 4\cdot\sin(45^0)}{\sqrt{3^2+4^2}}$$

$\approx 0,99$

b) $z = \sqrt{xy}$ $\quad P_0\,(2,2,2) \quad \alpha = \frac{2}{3}\pi$

$$z_x = \frac{1}{2}(xy)^{-\frac{1}{2}} \cdot y$$

$$z_y = \frac{1}{2}(xy)^{-\frac{1}{2}} \cdot x$$

$$\frac{\partial z}{\partial \vec{a}} = z_x \cdot \cos\alpha + z_y \cdot \sin\alpha$$

$$= \frac{y \cdot \cos\alpha + x \sin\alpha}{2\sqrt{xy}}$$

$$= \frac{2 \cdot \cos\left(\frac{2}{3}\pi\right) + 2 \cdot \sin\left(\frac{2}{3}\pi\right)}{2 \cdot \sqrt{2 \cdot 2}}$$

$$= \frac{2 \cdot \left(-\frac{1}{2}\right) + 2 \cdot \frac{\sqrt{3}}{2}}{2 \cdot 2}$$

$$= \frac{\sqrt{3} - 1}{4}$$

5.) $T(x;y;z) = xy + 2yz \qquad (x;y;z) \in B \subseteq R^3$

$P_0\ (2;1;-1) \quad \vec{e_s} = (1;1;1)^T$

$T_z = 2y$

$T_y = x + 2z$

$T_z = y$

$$\frac{\partial z}{\partial \vec{a}} = \frac{1}{\sqrt{a_1^2 + a_2^2 + a_3^2}} \cdot (a_1 T_x + a_2 T_y + a_3 T_z)$$

$$= \frac{1}{\sqrt{1^2+1^2+1^2}} \cdot \left\{ 1 \cdot 1 + 1 \cdot [2 + 2 \cdot (-1)] + 1 \cdot 2 \cdot 1 \right\}$$

$$= \sqrt{3}$$

6.)

a) $z = 3x^2y + 4y^3 - 3x^2 - 12y^2 + 1$

$z_x = 6xy - 6x$

$z_y = 3x^2 + 12y^2 - 24y$

$z_{xx} = 6y - 6$

$z_{yy} = 24y - 24$

$z_{xy} = 6x = z_{yx}$

$0 \stackrel{!}{=} 6xy - 6x = 6x(y-1) \quad \rightsquigarrow x_1 = 0$

$\rightsquigarrow y_1 = 1$

$0 \stackrel{!}{=} 3x^2 + 12y^2 - 24y$

$0 \overset{!}{=} 3x^2 + 12y^2 - 24y \qquad x_1 = 0 \qquad y_1 = 1$

$0 = 3 \cdot 0^2 + 12y^2 - 24y$

$0 = 12y^2 - 24y = y(12y - 24) \rightsquigarrow y_{s1} = 0$

$y_{s2} = 2$

$x_{s1} = 0 \qquad y_{s1} = 0$

$x_{s2} = 0 \qquad y_{s2} = 2$

$0 = 3x^2 + 12 \cdot 1^2 - 24 \cdot 1$

$0 = 3x^2 - 12$

$4 = x^2$

$x_{s3,4} = \pm 2$

$x_{s3} = 2 \qquad y_{s3} = 1$

$x_{s4} = -2 \qquad y_{s4} = 1$

$$H = \begin{pmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{pmatrix} = \begin{pmatrix} 6y - 6 & 6x \\ 6x & 24y - 24 \end{pmatrix}$$

$$\det(H) = (6y-6)(24y-24) - (6x)^2$$

$\det(H_1) = (6 \cdot 0 - 6)(24 \cdot 0 - 24) - (6 \cdot 0)^2 = 144 \rightsquigarrow EP$

$\det(H_2) = (6 \cdot 2 - 6)(24 \cdot 2 - 24) - (6 \cdot 0)^2 = 144 \rightsquigarrow EP$

$\det(H_3) = (6 \cdot 1 - 6)(24 \cdot 1 - 24) - (6 \cdot 2)^2 = -144 \rightsquigarrow SP$

$\det(H_4) = (6 \cdot 1 - 6)(24 \cdot 1 - 24) - (6 \cdot (-2))^2 = -144 \rightsquigarrow SP$

$z_{xx}(x=0; y=0) = 6\cdot 0-6 = -6 < 0 \rightsquigarrow$ lok. Maximum

$z_{xx}(x=0; y=2) = 6\cdot 2-6 = 6 > 0 \rightsquigarrow$ lok. Minimum

$z_1 = 1$

$z_2 = 32-48+1 = -15$

$z_3 = -12+4-12-12+1 = -7$

$z_4 = -7$

$P_1(0; 0; 1) \rightarrow$ Maximum

$P_2(0; 2; -15) \rightarrow$ Minimum

$P_3(2; 1; -7) \rightarrow$ Sattelpunkt

$P_4(-2; 1; -7) \rightarrow$ Sattelpunkt